13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

REASONING WITH RISK: A SURVIVAL GUIDE

Manfred Borovcnik & Ramesh Kapadia

University of Klagenfurt

This paper draws on several decades of research to discuss the nature of chance and probability, and probabilistic thinking, within the theme of risk. It introduces the terminology of APT (a priori theory), FQT (frequentist theory) and SJT subjectivist theory. We discuss current examples of risk, especially relating to health issues to illustrate another theme that more data can lead to more uncertainty, which differs from probability as a concept. We end by setting some goals for the future.

INTRODUCTION AND TERMINOLOGY

The title of this paper alludes to the origins of probability in the midst of time. In particular, the paper deals with the first theme of the ICME Study Group on the nature of chance and probability, with some thoughts about the second theme on probabilistic thinking. Our views are based on our life-long working experience in the field and so we will start with a historical commentary about both the subject as well as its teaching. Some of the reasons that teaching probability, especially at the early stages, is not easy, may well become clear by reference to the historical sources and the emergence of the concept of probability. The aim is to present an overview with the key reference to our recent contribution to Chernoff and Sriraman (2014), where many other references can be found. The references to risk are about modern approaches where probability is strongly promoted in the school curriculum in order to prepare children better for the world.

We will use the terminology of APT (a priori theory), FQT (frequentist theory) and SJT (subjectivist theory) to delineate the key strands, which are important for the teaching of probability from the early introduction to the subject. This approach was originally advocated by the Schools Council Project on Statistical Education about forty years ago (1980): the first booklet on probability for Year 7 (grade 6) began with ideas of fairness arising from games and introduced all three approaches to probability. We hope that our recently devised terminology of APT, FQT, and SJT is adopted across all areas of research but would be willing to discuss alternatives by others who are involved in this important field of research. The most comprehensive description of these terms is given in Borovcnik and Kapadia (2014a; 2014b), the opening chapters of Chernoff and Sriraman (2014).

In this paper, most of the sources used are in the English language; we hope that other international researchers will be able to supply key references in other languages, including Spanish, Italian and Portuguese. We assert that such a terminology encapsulates in a direct, dynamic way, the key issues, which are under discussion within the field of probability education. This also stresses the pluralistic perspectives necessary within the discipline. If adopted, this will stress the need for multiple approaches to the teaching of probability.

Risk has been a feature of life ever since man roamed the planet and ways to deal with it have been sought to reduce unwanted outcomes. Many rituals have been devised, which are often linked to religion; indeed it was often prophets and priests who were at the forefront to use chance in their rituals. Each day, the decisions we make are based on informal assessments of risk. Even those with good training in mathematics rarely use formal approaches in daily decisions. This key element needs to be remembered when teaching probability in school. An important reference to these origins of probability is by Florence David (1962).

Chance and Probability

Both the words ‘chance’ and ‘probability’ are part of everyday language in English and other languages. The word ‘chance’ has many connotations and is used in figurative senses with regards to various events such as the weather and sporting results. What is the chance of rain tomorrow or Germany winning Euro 2020? The answers are based on the terminology of words such as likely, unlikely, and probable and improbable. These words have also developed a meaning in legal cases where certainty is rarely possible. Thus every child or adult hears these words regularly in a variety of situations and contexts. Yet, in the mathematics class, much more precision is required and demanded. There are links so that *more than likely* would signify a chance of over 50%; however, the value assigned to the word *probable* is less clear, but usually higher than for ‘more than likely’. Eventually, the scale from 0 to 1 is used, yet the lack of link to everyday language continues to be troubling. Apart from logical examples there are few cases of certain or impossible events.

The difficulties are exacerbated when, for some event in question, the result is seen as a desirable outcome (Austria winning Euro 2020) or an undesirable outcome (rain). Such issues should feature regularly in class discussion, yet this is rare in reality. This could be one of the reasons why probability is deceptively difficult, unlike the concept of numbers, for example. For exposure to the mathematical aspects of numbers, it is essential to start very early in a child’s learning life. Probability figures much later in the school curriculum, when many prior intuitions have already been formed and are then hard to change. A notorious example for this “development” is when children exclaim that in dice rolling it is harder to get a six than other numbers because of the rules of many board games where dice are used. Indeed, the mind may well think that it *is* harder to get a six than any other selected number, because the brain does not have a mechanism for remembering how often a 4 really did occur – unlike the occasional frustration felt when a six takes a long time to appear (and a four is rarely needed likewise), or the pleasure of getting a six on the first throw.

Yet, the mathematical story of probability begins with Fermat, Pascal (Pascal and Fermat, 1654) and Laplace leading to the first formulation of a definition of probability, which was set in terms of APT or equal chances for a set of outcomes such as throwing dice or tossing coins, which is now universally accepted (Laplace, 1812). Even at that time there were references to FQT or frequentist probability: the first of the problems investigated is reputed to relate to gambling experiences of events which were thought to be equally likely using proportional reasoning. Indeed, proportional reasoning, which is found to be more complex than one might think, is a key component of probability.

The third strand of probability is due to de Finetti (1937), from a formal perspective, though its story actually begins in ancient history with man’s daily experience of the forces of nature on daily life. This subtle underpinning, with hopes and desires for the future, is SJT – the intuitive or subjectivist approach to probability. Crucially, it is not subjective, but *subjectivist* probability, since there are implicit rules (axioms) to be followed. For de Finetti, coherence is the key axiom; easier formulations do exist and should be developed for use in teaching. In practice, coherence is quite a complicated axiom and even hard for probability experts to use (it is especially hard to transfer this principle to their daily life). An example is the long (and often emotional) exchanges on the famous Monty Hall problem (Borovcnik, 2012); there are also many examples in the Nobel Prize winning work of Daniel Kahneman, discussed below.

Probabilistic Thinking

The key axiomatic formulation leading to the acceptance of probability by mathematicians was developed by Kolmogorov early in the last century (Kolmogorov, 1933); though philosophical debates about the meaning of probability continued, including the startling declaration by de Finetti (1937) that (to underline its importance, de Finetti put it exactly in the layout reproduced here)

PROBABILTY DOES NOT EXIST.

The use of capital letters emphasises the deep explorations of the philosophical approach to probability undertaken by de Finetti, as discussed in Borovcnik and Kapadia (2014a). Basically, probability is viewed by de Finetti as personal judgement, which may – in a modern view – also be connected to a model or construct relating to an event, it is *not* a property of the event, in the way that the mass of an object is. The way that probability underpins statistics, was developed in the last century, notably by Neyman and Pearson, with the deeper exploration of randomness (Neyman & Pearson, 1928; Neyman, 1937). Our view in the opening chapter in Chernoff and Sriraman (2014) is that probability is key to developing ideas in statistics, from the earliest stages of collecting data. We also firmly advocate that the three approaches of APT, FQT, and SJT have to be presented collectively when teaching ideas of probability in the curriculum from primary school. This view is based on the psychological research conducted over the last seven decades, which is briefly summarised in the next section.

Psychological Research

The earliest substantial psychological research, which is unlikely to be replicated in terms of its comprehensiveness, scope and time spent was conducted by Piaget and Inhelder (1951). Their work was linked to the APT theory of probability. Their tasks, based on traditional devices such as urns and spinners, were sometimes quite complex and undertaken with a small number of children – but over an extended period. They found out that young children could not distinguish between certainty and uncertainty and so asserted that probability should only be taught to older pupils that have reached already at formal stages of reasoning. The next major empirical educational work was by Fischbein (1975), who based his approach on FQT and SJT with young children. He felt that the work of Piaget and Inhelder had two essential deficiencies: his own work showed that young children do indeed distinguish between certainty and uncertainty (SJT), particularly where a high level of verbal understanding is not needed. Fischbein also supported the positive value of the learning process. More details can be found in Hawkins and Kapadia (1984)

The next major development in psychological research was by Kahneman, Slovic and Tversky (1982) whose extensive work with students and adults showed serious deficiencies in probabilistic understanding. After the tragic early death of Tversky, Kahneman has continued the work (e.g., Kahneman, 2011), for which he has been awarded the Nobel Prize in economics. The research has uncovered a number of fallacies and generated much subsequent educational research such as by Huerta (2014), Chiesi & Primi (2014), and Kapadia (2013). The initial work by Kahneman showed the difficulty people have in applying Bayes' formula and using base-rate information. This has been confirmed in subsequent research in the educational field, e.g. by Zapata Cardona (2008), or Chernoff (2012).

Our conclusion is that Bayes' formula is important but difficult and so can only be taught to higher ability pupils and students (perhaps the top 10%); Bayes' formula also requires much practice to develop an intuitive understanding, as required in similarly difficult topics such as solving quadratic equations or proving the Euclidean circle theorems (such as the properties of cyclic quadrilaterals or intersecting chords) in geometry. In practice, Bayes' formula is contrary to the ubiquitous FQT and difficult even for experienced teachers. One is applying a model which poses its own difficulties. Since probability is a minor part of the mathematics curriculum, students rarely get sufficient practice and this is reflected in difficulties found even at university level amongst mathematics undergraduates. This is partly because it is badly taught but also because people revert to prior intuitions and beliefs, which can prove unreliable; rules such as favourable divided by possible cases are useful but also have key limitations, which are rarely explored in mathematics classes. A lot of practice is needed when dealing with probabilistic information *before* calculating probabilities. As an example, though we may know that it is irrational (from a money perspective only) to buy lottery tickets, we still do in that the Gods may favour us: we would estimate that over 90% of participants of ICME have bought a lottery or raffle ticket in the last four years.

Another important strand of relevant psychological research has been developed by Gigerenzer and his school in Berlin (Gigerenzer, 2003, Martignon, 2014). Martignon and Krauss (2009) have also worked with young children. Their findings relate to ideas about 'whole number' and 'proportional' reasoning within probability, as well as the relative merits and the effectiveness of 'fast and frugal' methods.

The ideas from psychological research have been used in developing educational materials, as well as in education research. An approach to synthesise these elements was first undertaken by Kapadia and Borovcnik (1991). The prefaces, commentaries and summaries in Chernoff and Sriraman (2014) build on this synthesis.

Surviving Risk

Perhaps the major point to make is that risk is far lower now than at any point in the past. Tragedies are far less common, as shown in the marked rise in life-span and general well-being, with dramatic falls in early mortality. All over the world there is much less risk of disease and poor health, despite the constant coverage of such events in the media. These are major positive developments.

Nevertheless new issues have emerged which relate to probability and risk, which we now discuss, within the context of improving probability education.

In terms of the real-life examples, there are several current topical issues from this century. These include approaches to flu epidemics, the problems from ash clouds, and BSE. The underlying issues are quite deep and complex, which may be seen from the analysis in Borovcnik (2015). One aspect involves the notions of false positives and false negatives, which we will discuss in more detail in our oral presentation. The BSE issue is also quite subtle: it is hard to disbelieve experts, but one should always remember that they have their own values and prejudices, not least relating to their self-perceived expertise (Borovcnik & Kapadia, 2011a; 2011b). The pharmacological and health industries have powerful interests, which may not always align with 'public good'. Two more trivial but illuminating examples are the extensive discussions about the Monty Hall paradox (choosing a winning box from three after extra information; see Borovcnik, 2013) and the famous correspondence on Division of Stakes between Pascal and Fermat, with the notion of imaginary games to calculate probability accurately (see Borovcnik & Kapadia, 2014a).

Returning to flu epidemics, scares have arisen over the last few centuries. The scale of concern has increased markedly in the last few decades with increasing travel and enhanced communication, but it is not clear that the responses have improved in terms of a better understanding of probability. Indeed, it could be argued that a poor understanding of probability has led to a worse response politically, certainly in terms of the extra money spent, for example in stock-piling medicines which turn out not to be used. This trend is perhaps encouraged by pharmacological companies, who can make large profits from 'scare-mongering'. The risks to life can be greatly exaggerated in the media, leading to public fears and concerns, with over-cautious responses from politicians. Even if the risk is very low, no politician can afford to 'do nothing' as his/her reputation would be irrevocably damaged if the highly unlikely event does indeed occur: a price of democracy (which may well be worthwhile anyway). However, our main concern here is whether the underlying probability is well understood, and how a rational perception of the context and the evaluation of risks and probabilities can be improved.

Other examples are the ash-cloud in 2010 and BSE early this century. Flight of aeroplanes was stopped and only resumed after great commercial pressure was applied by airlines. The BSE controversy in the United Kingdom led to mass culls, which some now believe were unnecessary. It might be that all positive tested BSE cattle were actually false positives (see Dubben & Beck-Bornholdt, 2010, p. 64)

These real-life dilemmas about everyday decisions show that whilst the conventional APT and FQT approaches to probability should form the backbone of the concept of probability, ignoring SJT only stores up problems for the future in applying mathematical skills in daily life and in applications. The approach we advocate is the use of paradoxes and fallacies in teaching probability, as exemplified in the second chapter of Chernoff and Sriraman (2014). In the verbal presentation, some examples will be shared, especially where FQT does not work, as well as famous historical examples that have led to new paradigms of reasoning. These examples are pertinent in terms of statistical versus probabilistic knowledge and reasoning.

The two aspects are closely linked since, for all data, the question must be posed about its representativeness and replicability when applying tests of inference. This theme is developed in the first chapter of the book by Chernoff and Sriraman (2014), while the approach to probability through paradoxes and puzzles is presented in the second chapter. Modelling is also explored. Our approach follows on the tradition of Felix Klein (1908) and his advocacy of strong links between the historical, theoretical and philosophical ideas of mathematics and mathematics education. Now substantial research in psychology is a new component as it provides different insights into cognitive development. Probability also offers a powerful vehicle where theoretical ideas can be developed into applications, especially risk.

Future goals

We end with some thoughts on future research in probability education. This returns to the ten assertions made at the end of our book Kapadia and Borovcnik (1991). The first three statements relate to the (personal, incomplete, and biased) ways people process information: much more research has been done in this area and we would stand by the assertions, though perhaps stress positive aspects of using memorable events and recognise that it is difficult to take a comprehensive approach daily, even for statisticians. Sir David Spiegelhalter (Spiegelhalter & Gage, 2015) has undertaken much research on the next three assertions about people's handling of very low and very high probabilities: this should be disseminated more widely and be explicitly included in the school curriculum. Equal likelihood and related ways to transfer probability to unknown situations (statements 7 and 8) remain areas for further exploration and should feature strongly in class discussion, while examples of incoherent assignment of probabilities (including supra-additivity; statements 9 and 10) can help students avoid that trap.

Goals for the future of probability education should be promoted within the study group and we suggest two. One goal would be that better probability education improves the way that chance is dealt with by adults in their everyday life. We would also like the media to make more considered use of the underlying ideas of probability in reporting risk; birds may also fly.

With regards to teaching, students have to be offered a means to adapt from intuitive ways to handle probabilistic situations. This structure must go beyond singular examples in dealing with probability and could reveal extreme forces of archetypal patterns of behaviour, which are successful in fields other than probability.

Historical struggles provide a valuable orientation. While empirical research about how people think and how successful teaching programmes have been are helpful to improve teaching plans, we should not to lose sight of key concepts and strategies from the past. One key lesson from history is that probability has always been a *pluralistic* concept and has drawn its meaning from the interplay of its interpretations. Simplification is a basic ingredient of teaching. However, sometimes it may undermine the complexities for certain concepts, which we will discuss in our presentation. The rather narrow focus on the frequency interpretation ignores some facets of probability. Simulation provides a *solving technique* but does not help to *model* a situation and discourages probabilistic thinking.

Conclusion

In conclusion, we repeat our plea above that multiple approaches to probability including APT, FQT and SJT are essential in the teaching of probability. We have discussed probabilistic thinking in the context of risk. We have suggested some goals for probability education in order to set an agenda over the next four years before ICME 14.

References

Borovcnik, M. (2012). Multiple perspectives on the concept of conditional probability. *Avances de Investigación en Didactica de la Matemática, 1*(2) 5-27. Online: www.aiem.es/index.php/aiem/issue/view/2.

Borovcnik, M. (2015): Risk and Decision Making: The "Logic" of Probability. *The Mathematics Enthusiast, 12*(1, 2 & 3), 113-139. Online: scholarworks.umt.edu/tme/.

Borovcnik, M., & Kapadia, R. (2011a). Risk in health: more information and more uncertainty. *Proceedings IASE Satellite Conference on "Statistics Education and Outreach"* (6 pp.). Voorburg: ISI. Online: iase-web.org/Conference_Proceedings.php?p=Stats_Education_and_Outreach_2011.

Borovcnik, M., & Kapadia, R. (2011b). Determinants of decision-making in risky situations. *Proceedings of the 58th World Statistics Congress* (6pp.). Voorburg: ISI. Online: 2011.isiproceedings.org/papers/950138.pdf.

Borovcnik, M., Kapadia, R. (2014a). A Historical and Philosophical Perspective on Probability. In E. J. Chernoff & B. Sriraman, *Probabilistic Thinking: Presenting plural perspectives. Advances in Mathematics Education, v. 7* (pp. 7-34). Berlin: Springer.

Borovcnik, M., Kapadia, R. (2014b). From Puzzles and Paradoxes to Concepts in Probability. In E. J. Chernoff & B. Sriraman, *Probabilistic Thinking: Presenting plural perspectives. Advances in Mathematics Education, v. 7* (pp. 35-73). Berlin: Springer.

Chernoff, E. J. (2012). Unintended relative likelihood comparisons. *Proceedings of Topic Study Group 11: Teaching and learning of probability*. 12th International Congress on Mathematics Education (ICME-12). Seoul, Korea. Online: www.icme12.org/sub/tsg/tsg_last_view.asp?tsg_param=11.

Chernoff, E. & Sriraman, B. (Eds.) (2014). *Probabilistic thinking: presenting plural perspectives*. *Advances in Mathematics Education, v. 7*. New York: Springer.

Chiesi & Primi (2014). The interplay among knowledge, cognitive abilities and thinking styles in probabilistic reasoning: a test of a model. In E. J. Chernoff & B. Sriraman, *Probabilistic Thinking: Presenting plural perspectives. Advances in Mathematics Education, v. 7* (pp. 195-214). Berlin: Springer.

David, F.N. (1962). *Games, gods and gambling*. London: Griffin.

Dubben, H.-H., & Beck-Bornholdt, H.-P. (2010). *Mit an Sicherheit grenzender Wahrscheinlichkeit. Logisches Denken und Zufall*. (With a probability coming close to certainty. Logic thinking and randomness.) Reinbek: Rowohlt.

Fermat, P. de, & Pascal, B. (1654). *Correspondence*, publ. 1679 in Toulouse; English translation in F.N. David (1962), *Games, gods and gambling* (pp. 229–253). London: Griffin.

Finetti, B. de (1937/1992). La prévision: ses lois logiques, ses sources subjectives. *Annales Institut Henri Poincaré, 7*, 1-68. Foresight: Its logical laws, its subjective sources (chapters 1-4 translated by H. E. Kyburg Jr.) in S. Kotz, & N. L. Johnson, *Breakthroughs in statistics. Volume I. Foundations and Basic Theory* (pp. 134-174). New York, Berlin: Springer.

Finetti, B. de (1974). *Theory of probability*. New York: Wiley (Translated by A. Machi, & A. Smith).

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht: Reidel.

Gigerenzer, G. (2002). *Calculated risks: how to know when numbers deceive you*. New York: Simon & Schuster.

Hawkins, A. E. & Kapadia, R. (1984). Children's conceptions of probability – A psychological and pedagogical review. *Educational Studies in Mathematics, 15*(4), 349-377.

Huerta, M. P. (2014). Researching conditional probability problem solving. In E. J. Chernoff & B. Sriraman, *Probabilistic Thinking: Presenting plural perspectives. Advances in Mathematics Education, v. 7* (pp. 613-639). Berlin: Springer.

Kahneman, D. (2011). *Thinking, fast and slow*. London: Penguin Books.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.

Kapadia, R. (2013). The role of statistical inference in teaching and achievement of students. *Proceedings of the 2013 World Statistics Congress* (pp. 1102-1107). Voorburg: International Statistical Institute. Online: http://2013.isiproceedings.org/Files/IPS112-P2-S.pdf.

Kapadia, R., & Borovcnik, M. (Eds.) (1991). *Chance encounters: probability in education*. Dordrecht: Kluwer.

Klein, F. (1908/2004). *Elementary mathematics from an advanced standpoint. Volume 1: Arithmetic, Algebra, Analysis, Volume 2: Geometry*. New York: Dover. Translation of *Elementare Mathematik von höherem Standpunkte aus*, 3rd ed. (1924-28). 1st ed. (1908). Leipzig: Teubner

Kolmogorov, A.N (1933/1956). *Foundations of the theory of probability*. New York: Chelsea.

Laplace, P.S. de (1812/1951). Essai philosophique sur les probabilités. *Journal de l'École Polytechnique VII/VIII*, 140-172. English translation: *A philosophical essay on probabilities* (extended version). New York: Dover.

Martignon, L. & Krauss, S. (2009). Hands-On Activities for Fourth Graders: A Tool Box for Decision-Making and Reckoning with Risk. *International Electronic Journal of Mathematics Education (IEJME), 4*(3), 227-258. Online: www.mathedujournal.com/issues.php?id=483.

Martignon, L. (2014). Fostering children's probabilistic reasoning and first elements of risk evaluation. In E. J. Chernoff & B. Sriraman, *Probabilistic Thinking: Presenting plural perspectives. Advances in Mathematics Education, v. 7* (pp. 149-160). Berlin: Springer.

Neyman, J. & Pearson, E. S. (1928/1967). On the use and interpretation of certain test criteria for purposes of statistical inference, part I and II. *Biometrika, 20 A*, 175-240; 263-294. Reprinted as *Joint statistical papers, Nr. 1 and 2*. Berkeley: University of California Press.

Neyman, J. (1937/1967). Outline of a theory of statistical estimation based on the classical theory of probability. *Philosophical Transactions of the Royal Society of London, A 236*, 333-380. Reprinted as *A selection of early statistical papers of Jerzy Neyman, Nr. 20*. Berkeley: University of California Press.

Piaget, J. & Inhelder, B. (1951). *La genèse de l'idée de hasard chez l'enfant*. Paris: Presses universitaires de France. English translation, *The origin of the idea of chance in children*. London: Routledge and Kegan Paul.

Schools Council Statistical Education Project 11-16 (1980). Teaching Statistics 11-16: *Statistics in your world*. (a handbook and a series of booklets). Slough: Foulsham.

Spiegelhalter, D. & Gage, J. (2015) What can education learn from real-world communication of risk and uncertainty? *The Mathematics Enthusiast, 12*(1, 2, & 3), 4-10. Online: scholarworks.umt.edu/tme/.

Zapata Cardona, L. (2008). How do teachers deal with the heuristic of representativeness? *Topic Study Group 13: Research and development in the teaching and learning of probability at the 11th International Congress on Mathematical Education (ICME 11)*. Online: www.ethikkommission-kaernten.at/ICME11/p22_ICME11_May16_LuciaZapata_mb.pdf.